Published by Institute of Physics Publishing for SISSA

Received: *December 19, 2005*
Accepted: *December 21, 2005*
Published: *January 12, 2006*

Long strings, anomaly cancellation, phase transitions, T-duality and locality in the 2d heterotic string

Nathan Seiberg

School of Natural Sciences, Institute for Advanced Study
Einstein Drive, Princeton, NJ 08540, U.S.A.
E-mail: seiberg@ias.edu

Abstract: We study the noncritical two-dimensional heterotic string. Long fundamental strings play a crucial role in the dynamics. They cancel anomalies and lead to phase transitions when the system is compactified on a Euclidean circle. A careful analysis of the gauge symmetries of the system uncovers new subtleties leading to modifications of the worldsheet results. The compactification on a Euclidean thermal circle is particularly interesting. It leads us to an incompatibility between T-duality (and its corresponding gauge symmetry) and locality.

Keywords: Superstring Vacua, Superstrings and Heterotic Strings, 2D Gravity.

JHEP01(2006)057

Contents

1. Introduction

Low dimensional theories are useful laboratories for studying simple phenomena which might have analogs in more physical dimensions. In string theory we expect the two-dimensional string to teach us about more generic phenomena in higher dimensions. The most studied two-dimensional string theories are the bosonic theory (for earlier reviews see, e.g. [1, 2]) and its type 0 relatives [3, 4]. These theories have known matrix model duals, which allow detailed nonperturbative studies. Other two-dimensional theories include the type II theories [5, 6], the heterotic theories [7 – 10] and various nonoriented and open string theories [11 – 13].

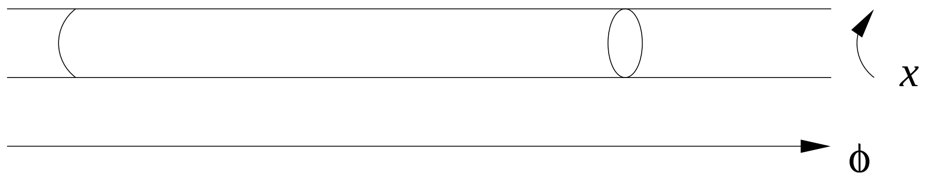

Figure 1: Our target space is an infinite cylinder. There is a linear dilaton direction ϕ and a compact direction x. We will study it in a Hamiltonian picture in two channels. The x-channel picture describes evolution along x with an infinite space direction ϕ. Here we evaluate a trace. The ϕ-channel picture describes evolution along ϕ from the weak coupling end $\phi \to -\infty$ to the strong coupling end $\phi \to +\infty$. Here the space is the circle parameterized by x.

In this note we will continue the investigation of the two-dimensional heterotic theory.[1] We will see that it shares some of the features of its bosonic, type 0 and type II counterparts, but it also exhibits many new phenomena. These include spacetime chirality and non-standard mechanisms for anomaly cancellation, new subtleties in applying worldsheet methods and new phase transitions.

As all two-dimensional string theories, our target space has a linear dilaton along the space direction ϕ with the weak coupling end at $\phi \to -\infty$. In addition, there is a time direction x which can have either Lorentzian or Euclidean signature. We will often regularize the ϕ direction in the weak coupling region and denote its volume by V; i.e.

$$-V < \phi \tag{1.1}$$

The Euclidean time coordinate x can be compactified on a circle of radius R with various twists. Then, our target space looks like an infinite cylinder (see figure 1). It has a noncompact coordinate ϕ and a compact coordinate x.

In the standard interpretation of this target space ϕ is a space coordinate and x is a Euclidean time coordinate. We will refer to this interpretation as the x-channel. An alternative interpretation is to view ϕ as Euclidean time and x as a compact space coordinate. Here we are interested in the Euclidean time evolution along ϕ from the weak coupling end at $\phi \to -\infty$ to the strong coupling end. We will refer to this picture as the ϕ-channel (see figure 1).

As in all other two-dimensional noncritical strings, the closed string spectrum of the theories include only a finite number of massless particles. The familiar Hagedorn density of states of higher dimensional string theory is absent here. Therefore, the target space theory appears to be similar to an ordinary field theory with a finite number of fields. However, in addition to the closed string spectrum the theory has infinite energy states associated with infinite long strings, which are stretched along the entire space direction ϕ. Such strings were studied in the two-dimensional bosonic and type 0 theories in [18, 19]. Long heterotic strings were recently studied in higher dimensions in [20]. We will see that the long heterotic strings have nontrivial degrees of freedom living on them, and therefore they behave differently than their bosonic and type 0 relatives.

[1] Other theories with somewhat similar features were studied in [14 – 17].

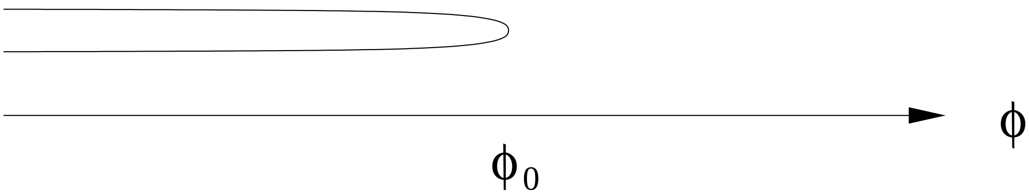

Figure 2: A pair of an infinite long string and an infinite long anti-string can be thought of as an infinite string coming in from the weak coupling end $\phi \to -\infty$, turning around at ϕ_0 and moving back to $\phi \to -\infty$.

The long strings couple to the two form field B. Therefore, a single long string leads to a B tadpole and does not satisfy the equations of motion. An infinite long string with the opposite orientation is a long anti-string and it couples to B with the opposite sign. Therefore, a pair of a long string and a long anti-string carries no B charge and can be added to the system. Such a pair can annihilate in the interior of the space and form a folded string which ends at a point ϕ_0 (see figure 2). Such configurations were analyzed in detail in the two-dimensional bosonic and type 0 theories in [18].

We will see that both a single long string and the pairs of string-anti-string excitations are important to the dynamics of the heterotic theory.

In section 2 we will review the closed string spectrum of the three different two-dimensional heterotic string theories in $\mathbb{R}^2$: HE, HO and THO. In section 3 we will quantize the long string, stressing the differences between its quantization and the more standard closed string quantization. These long strings will be used in section 4 to cancel spacetime anomalies. In section 5 we will start the discussion of compactifications, setting the notation and making general comments. Section 6 will introduce the ϕ-channel quantization and will explain the subtleties associated with gauge invariance. We will see that in some cases the worldsheet answers need to be modified.

In sections 7 and 8 we will discuss possible phase transitions in our system. In section 7 we will consider a transition associated with a massless complex fermion. In section 8 we will analyze a transition associated with a massless complex scalar. Here we will find a possible discrepancy between the ϕ-channel and the x-channel pictures.

Sections 9, 10 and 11 will be devoted to specific examples. In section 9 we will consider an untwisted circle compactification of our theories. In section 10 we will study compactifications twisted by worldsheet fermion number, and in section 11 we will discuss thermal compactifications; i.e. compactifications twisted by spacetime fermion number.

We will summarize and will further discuss our results in section 12.

An appendix will examine possible consequences of our results to the type 0/type II compactifications.

Our general discussion and especially sections 5–8, which precede the concrete examples, might appear somewhat abstract. Some readers might prefer to read them together with the example sections 9 - 11.

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2. Known heterotic theories

In addition to the worldsheet fields x and ϕ, our theory includes their $(0,1)$ superpartners which are right-moving fermions ψ_x and ψ_ϕ, and 24 left-moving fermions λ^I. Depending on the sum over spin structures three theories were identified [7, 9, 10] (two of them are analogous to their ten dimensional relatives and hence the terminology):

HO The gauge group is $Spin(24)$ and the spectrum includes massless "tachyons" in **24** of $Spin(24)$. Using standard notation their vertex operators are

$$T^I(k) = e^{-\varphi}\overline{\lambda}^I e^{ikx+(1-|k|)\phi} \qquad ; \qquad I = 1,\ldots,24 \tag{2.1}$$

HE The gauge group is $Spin(8)\times E_8$ and the spectrum includes massless "tachyons" in $\mathbf{8_v}$, left-moving fermions in $\mathbf{8_c}$ and right-moving fermions in $\mathbf{8_s}$ of $Spin(8)$. Their vertex operators are

$$\begin{aligned} T^i(k) &= e^{-\varphi}\overline{\lambda}^i e^{ikx+(1-|k|)\phi}; \qquad i=1,\ldots,8 \\ \Psi^\alpha &= e^{-\frac{1}{2}\varphi+i\frac{1}{2}H}\overline{S}^\alpha e^{ikx+(1-|k|)\phi}; \qquad \alpha=1,\ldots,8; \qquad k\geq 0 \\ \widetilde{\Psi}^{\dot\alpha} &= e^{-\frac{1}{2}\varphi-i\frac{1}{2}H}\overline{S}^{\dot\alpha} e^{ikx+(1-|k|)\phi}; \qquad \dot\alpha=1,\ldots,8; \qquad k\leq 0 \end{aligned} \tag{2.2}$$

Other versions of this theory which are related by $Spin(8)$ triality are physically equivalent.

THO This is a twisted (orbifold) version of HO. The gauge group is $Spin(24)$ and the spectrum includes right-moving fermions in **24** of $Spin(24)$

$$\widetilde{\Psi}^I(k) = e^{-\frac{1}{2}\varphi-i\frac{1}{2}H}\overline{\lambda}^I e^{ikx+(1-|k|)\phi}; \qquad I=1,\ldots,24; \qquad k\leq 0 \tag{2.3}$$

This theory appears to be anomalous, but we will see below that in fact it is consistent. A related equivalent theory is obtained from this theory by spacetime parity.

An interesting deformation of the HE and HO theories is a tachyon background[2] $\mu T^1(k=0) = \mu e^{-\varphi}\overline{\lambda}^1 e^\phi$. This deformation has the effect of breaking $Spin(8)\to Spin(7)$ and $Spin(24)\to Spin(23)$. In the zero ghost number picture the deformation of the (classical) worldsheet Lagrangian is

$$\mu\overline{\lambda}^1\psi_\phi e^\phi + \mu^2 e^{2\phi}\,. \tag{2.4}$$

Here, the second term of order μ^2 arises from imposing right-moving $(0,1)$ worldsheet supersymmetry.[3] Interestingly, even though the worldsheet theory has only $(0,1)$ supersymmetry, the interacting part of this Lagrangian has $(1,1)$ supersymmetry; it is the well studied $N=1$ super-Liouville theory [21–23]. Borrowing the results of the analysis of this

[2] As is common in the $c=1$ theory, depending on how we define the theory we might need another factor of ϕ in front of this expression.

[3] One way to see that is to consider the two $(0,1)$ superfields $\Phi = \phi + \theta\psi_\phi$ and $\overline{\Lambda}^1 = \overline{\lambda}^1 + \theta F^1$, where F^1 is an auxiliary field. Then the interaction term is $\mu\int d\theta\overline{\Lambda}^1 e^\Phi$. Equation (2.4) arises after writing it in components and integrating out the auxiliary field.

theory, we learn that it leads to a consistent string background. The deformation is negligible in the weak coupling region, and therefore it does not affect the extensive properties of the theory (those proportional to V). The "Liouville wall" (2.4) prevents the strings from propagating into the strong coupling region, and provides an effective coupling constant of order $\frac{1}{\mu}$.

In the $\mu \to \infty$ limit our theory is weakly coupled and can be analyzed using standard worldsheet methods. The sphere contribution is of the form $a_0\mu^2 \log \mu$, the torus is $a_1 \log \mu$ and higher corder corrections are suppressed by powers of $\frac{1}{\mu^2}$. Experience in other two-dimensional string theories suggests that the zero point function (logarithm of the second quantized functional integral) is of the form

$$\begin{aligned} \log \mathcal{Z} &= -(a_0\mu^2 + a_1)V + f(\mu) + \mathcal{O}\left(\frac{1}{V}\right) \\ &= a_0\mu^2(\log \mu - V) + a_1(\log \mu - V) + \sum_{n=2}^{\infty} a_n \frac{1}{\mu^{2n-2}} + \mathcal{O}\left(\frac{1}{V}\right) \end{aligned} \tag{2.5}$$

with $f(\mu)$ a smooth function of μ which depends on the strong coupling region of the theory. In most of our discussion we will be interested in the value of a_1. It can be thought of in two different ways. First, it is the extensive part of $\log \mathcal{Z}$ which can be calculated with $\mu = 0$. Alternatively, it is the coefficient of $\log \mu$ in $f(\mu)$.

There is no way to add such a tachyon condensate with a parameter μ in the THO theory, and therefore the behavior of this theory is sensitive to the physics in the strong coupling region.

3. Quantization of long strings

In this section we will study the quantization of long strings. This quantization is similar to that of the short closed strings but differs from it in a few crucial details.

We start by considering a single long string. One way to quantize it is to choose a static gauge where ϕ and x are identified with the worldsheet coordinates. This eliminates them as worldsheet superfields, and therefore removes also their fermionic super-partners ψ_ϕ and ψ_x. Equivalently, we can follow the standard lightcone gauge fixing procedure. Unlike standard closed strings, the long string is not subject to periodic boundary conditions, and therefore the zero modes of x and ϕ are also removed.

Another difference between this quantization and the quantization of the closed strings is that the $L_0 = \overline{L}_0$ constraint should not be imposed. This constraint is associated with the periodic boundary conditions of the closed string which are absent here.

Finally, since the string is infinite, we do not have NS and R sectors for its left-moving fermions. Similarly, we do not perform a GSO projection by summing over the spin structures in the time direction.

To summarize, the spectrum of the long string includes 24 left-moving fermions. Clearly, a long anti-string is a long string with the opposite orientation and its spectrum includes 24 right-moving fermions.

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Note that the spectrum of the long string is the same in all our three theories HE, HO and THO.

Now let us consider a pair of a long string and a long anti-string. This pair can form, as in figure 2, an infinite folded string which ends at ϕ_0. We would like to make several comments about this string.

First, because of its infinite length, its energy is infinite. With a finite cutoff $\phi > -V$ the energy of the string is proportional to $V + \phi_0$. The ϕ_0 dependence leads to a force which pushes the string toward the weak coupling end, $\phi_0 \to -\infty$. This dynamics was studied in the bosonic and type 0 noncritical string in [18]. Unlike the situation there, in our case this infinite folded string has oscillators living on it. Below we will examine their dynamics and will see that these oscillators can lead to important consequences.

Second, we would like to compare the infinite folded string to a long closed string. Locally, a very long closed string is similar to the folded string. The difference between them is that the folded string is infinite, or if we regularize the space at $\phi > -V$, its end is glued to the boundary at V. This fact means that unlike the long, but finite closed string, the folded string is not invariant under rigid shifts of its space coordinate σ. Therefore, the closed string has to satisfy the $L_0 = \overline{L}_0$ constraint, but the long folded string does not have to satisfy this relation. This is the reason the fermionic oscillators lead to physical excitations on the long folded string, but not on the closed string.

4. Anomaly cancellation with long strings

The THO theory has gauge and gravitational anomalies. We propose that these anomalies can be cancelled by adding a single long string. This anomaly cancellation can be understood in two different ways.

The simplest way to understand it is to note that the anomalies of the 24 spacetime right-moving fermions is cancelled by the anomaly of the 24 fermions of the long string. These left-moving worldsheet fermions become left-moving spacetime fermions and hence they cancel the anomaly.

An alternative way to understand it is the following. We can attempt to cancel the anomaly using the Green-Schwarz mechanism. The Green-Schwarz mechanism involves two facts. First, the gauge and gravitational gauge transformations affect the B field. This is true in any theory and follows simply from the analysis of the two-dimensional worldsheet theory. The second element is the Green-Schwarz term. In two dimensions it is simply $\int B$. Clearly, with such a term the anomalies are cancelled. However, with such a term the B equation of motion cannot be satisfied – there is a nonzero B tadpole. This can be fixed by adding a long fundamental string.

More generally, if we have W long strings and $\overline{W}$ anti-long strings, the B tadpole condition is $W = \overline{W} + 1$ in the THO theory ($W = \overline{W} - 1$ in the parity transform of the THO theory) and $W = \overline{W}$ in the HE and HO theories.

It is amusing to compare the effect of the Green-Schwarz term in different dimensions. As the number of dimensions is reduced the term has fewer derivatives and its effect is more dramatic. In four dimensions it leads to the Higgs mechanism [24, 25]. Sometime it also leads to tachyonic masses to some of the closed string modes and the need to shift their expectation values. In two dimensions it leads to a B tadpole and the necessity to add fundamental strings.

Finally, we would like to point out that this anomaly cancellation mechanism is similar to other examples where a space filling brane is needed in order to cancel anomalies. A well known example is the IIB theory in ten dimensions where the anomaly of the theory with an orientifold is cancelled by adding D-branes [26]. Two dimensional examples which are similar to ours appeared in four-fold compactifications of the ten dimensional type IIA theory [27].

5. Compactifications, preliminaries

In this section we will begin the discussion of the compactifications of these theories on various twisted circles of radius R. All the compactifications of the heterotic theory are on the same moduli space of vacua, which is parameterized by the radius R and the Wilson line around the circle [9]. We are going to limit ourselves to compactifications which preserve the gauge symmetry. This means that the Wilson line is in the center of the gauge group.

In all our examples the closed string vertex operators have the form

$$\begin{aligned} T_r &= e^{-\varphi}\overline{\mathcal{O}}_r\left(\frac{1}{2}+NW\right)V_{N,W} \\ \Psi_r &= e^{-\frac{1}{2}\varphi+i\frac{1}{2}H}\overline{\mathcal{O}}_r\left(\frac{1}{2}+NW\right)V_{N,W}\ , \qquad p_R\geq 0 \\ \widetilde{\Psi}_r &= e^{-\frac{1}{2}\varphi-i\frac{1}{2}H}\overline{\mathcal{O}}_r\left(\frac{1}{2}+NW\right)V_{N,W}\ , \qquad p_R\leq 0 \end{aligned} \tag{5.1}$$

with

$$\begin{aligned} V_{N,W} &= e^{i\frac{N}{R}(x+\overline{x})+i\frac{W}{2}R(x-\overline{x})+(1-|p_R|)\phi} \\ p_R &= \frac{N}{R}+\frac{WR}{2}\,. \end{aligned} \tag{5.2}$$

Here we have separated the coordinate x to its worldsheet left and right-moving components $\overline{x}$ and x.[4] $\overline{\mathcal{O}}_r(\overline{\Delta})$ are operators in the conjugacy class $r=0,V,S,C$ of $Spin(24)$ in HO and THO and of $Spin(8)\times E_8$ in HE with dimension $\overline{\Delta}$. Values of (N,W) which do not satisfy the inequalities of p_R, or lead to $\overline{\Delta}$ with no $\mathcal{O}(\overline{\Delta})$ do not correspond to physical operators.

The different theories and the different compactifications differ by the sets of (r,N,W) in (5.1). The values of W are even or odd integers while N are either integer or half integer. Below we will parameterize them in terms of integers as

$$N(n,w), \qquad W(n,w), \qquad \text{with} \qquad n,w\in\mathbb{Z} \tag{5.3}$$

We will be interested in the extensive part of the logarithm of the second quantized functional integral

$$\log\left(\mathcal{Z}(R)\right)=V\Gamma(R)+\mathcal{O}(1)\,, \tag{5.4}$$

[4]We use the same notation x for the right-moving part and the full field, hoping that this will not lead to a confusion.

where V is the volume. Usually it is obtained in the one loop approximation.

In all our examples we have locally

$$\Gamma(R) = aR + \frac{b}{R} \tag{5.5}$$

The first term aR is identified with minus the vacuum energy density. It is not calculable in field theory, but is finite in string theory. a is independent of the details of the compactification; i.e. different compactifications of the same theory have the same the large R limit of $\Gamma(R)$. Usually, the string computation of a is based on the torus amplitude with noncompact x. We denote this value of a by a_{closed} because it is associated with closed strings. The distinction between the true value of a and the torus answer a_{closed} will be clear below.

The second term $\frac{b}{R}$ probes the spectrum of the theory. It is also usually derived by a torus computation for large R. We will denote its value by b_{closed}. For sufficiently large R the constant b_{closed} can also be calculated using a one loop field theory calculation based on the $W = 0$ modes in (5.1) [28]. One way to perform this calculation is to sum (using ζ-function regularization)

$$\frac{b_{closed}}{R} = -\frac{1}{2} \sum_{\substack{W=0 \text{ terms} \\ \text{in (5.1)}}} (-1)^F |p_R| \tag{5.6}$$

with F the spacetime fermion number. Note that since we limit ourselves to $W = 0$, $p_R = \frac{N}{R}$.

We conclude that for sufficiently large R

$$\Gamma_{worldsheet}(\text{large } R) = \Gamma_{closed} = a_{closed}R + \frac{b_{closed}}{R} \tag{5.7}$$

However, the careful analysis in [9] showed that for sufficiently small R the torus amplitude $\Gamma_{worldsheet}$ does not need to agree with Γ_{closed}. In fact, it was shown that in some examples $\Gamma_{worldsheet}$ exhibits a phase transition, while Γ_{closed} is smooth. Furthermore, as we will discuss below, even for large R where $\Gamma_{worldsheet} = \Gamma_{closed}$ the correct value of Γ can be different!

The theory with compact x has two shift symmetries corresponding to the two conserved charges N and W. The deformation (2.4) is the only possible deformation preserving both N and W. In some of our compactifications the operator (2.4) is twisted, and then this deformation is not present. Other deformations break either N, or W, or both. For example, using $T_{r=V}$ in (5.1) we sometimes have the operator

$$\mu_{sl}\overline{\lambda}\psi \left(V_{N=0,W=1} + V_{N=0,W=-1}\right) + \dots \tag{5.8}$$

where μ_{sl} is the coupling constant, $\overline{\lambda}$ is one of the left-moving fermions, ψ is an appropriate linear combination of ψ_x and ψ_ϕ and the ellipses represent $\mathcal{O}(\mu_{sl}^2)$ corrections which are needed for $(0,1)$ worldsheet supersymmetry. This worldsheet theory is the same as the more familiar $N = 1$ super-sine-Liouville theory. Clearly, translation invariance is preserved, but translation around the dual circle, or W conservation, is no longer a symmetry.

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Another interesting deformation is by an operator constructed out of $T_{r=0}$ in (5.1). The simplest such deformation is

$$\nu\psi\left(V_{N=\frac{1}{2},W=-1}+V_{N=-\frac{1}{2},W=1}\right)+\cdots \tag{5.9}$$

where ν is the coupling constant. Like the sine-Liouville interaction (5.8), this operator depends on R. It breaks both N and W conservation but leaves one linear combination of them unbroken. Unlike (5.8), the operator (5.9) preserves T-duality under $R\to\frac{1}{R}$. Although we will not analyze the detailed dynamics of the theory deformed by this operator, we point out that its x dependent part is relevant for $\sqrt{2}-1<R<\sqrt{2}+1$, and therefore in this range we expect its dynamics to be most interesting. Also, at the selfdual point $R=1$ the operator is massless — its Liouville dressing is similar to that of (2.4). At that point the operator (5.9) is independent of the right-moving x and its dependence on $\overline{x}$ is through $e^{i\overline{x}}$, which can be thought of as a bosonized fermion. Therefore, at this point this operator is identical to (2.4).

6. Modifying the vacuum — changes to the worldsheet description

We will now examine the canonical quantization of our second quantized theory in the ϕ-channel, and will focus on the asymptotic weak coupling region. Viewing x as a space coordinate, and reducing on that circle, our system becomes a quantum mechanical system with ϕ being Euclidean time.

We regularize the ϕ direction, $\phi\in(-V,0)$. The boundary conditions at the two ends $\phi=-V,0$ can be thought of as boundary states $|s_V\rangle$ and $|s_0\rangle$. Then, our amplitude is the euclidean time evolution from $\phi=-V$ to $\phi=0$

$$\mathcal{Z}=\langle s_0|e^{-VH_\phi}|s_V\rangle \tag{6.1}$$

where H_ϕ is the ϕ-channel Hamiltonian. The two boundary states $|s_V\rangle$ and $|s_0\rangle$, which appear symmetrically in this evolution, are conceptually different. The state $|s_0\rangle$ summarizes the behavior of the system in the strong coupling region. It is determined by the dynamics of the theory. If the theory is not deformed by an interaction like (2.4) or others, $|s_0\rangle$ cannot be determined without a nonperturbative definition of the theory. However, if a term like (2.4) is present, the state $|s_0\rangle$ can be determined in the $\mu\to\infty$ limit by a tree level calculation. The state $|s_V\rangle$ is somewhat arbitrary. It determines the boundary conditions at the weak coupling end and through these boundary conditions it leads to different observables.

One choice for the state $|s_V\rangle$ is the ground state. Other choices are obtained by acting on it with creation operators. We identify these creation operators with the physical vertex operators (5.1). Their energy is $|p_R|$.[5] We conclude that different correlation functions of

[5]The branch of the ϕ dressing is such that the field configuration associated with each operator diverges at the weak coupling end $\phi\to-\infty$ [29, 30]. This choice of branch corresponds in the ϕ-channel to a creation operator. The opposite branch which vanishes as $\phi\to-\infty$ is an annihilation operator. One way to see that is to note that these operators raise the energy of the ϕ-channel state. We will see that in more detail below.

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vertex operators correspond to different states at the weak coupling end $|s_V\rangle$, and lead to different physical observables.

For large R the low lying quanta are the $W = 0$ modes in (5.1). They are obtained by reducing the spectrum of the theory with noncompact x on a circle with appropriate boundary conditions. In performing this dimensional reduction we should remember that our theory is a gauge theory. The gauge fields, the graviton and the B field of the two-dimensional theory lead to three kinds of gauge fields in the ϕ-channel theory $\oint dx A_\phi$, $\oint dx B_{x\phi}$ and $\oint dx G_{x\phi}$. We should now address the effect of these gauge symmetries.

The easiest gauge symmetry to address is the one associated with $B_{x\phi}$. It imposes conservation of the charge W. There are no closed strings with nonzero W in our low energy approximation at large R, so naively it has no effect. However, as we have seen above, the anomaly of the THO theory leads to a B tadpole and the necessity to add a long string. In our quantum mechanical system this means that the vacuum carries nonzero W. Clearly, $W = \pm 1$. Instead of directly determining its sign, we will rely on the consistency of the calculations below to set $W_{vacuum} = -1$. Then the $B_{x\phi}$ gauge invariance forces us to add a closed string vertex operator from (5.1) with $W = 1$ in order to cancel this charge. The addition of this vertex operator is the ϕ-channel way of describing the addition of a single long string in the x-channel.

A more subtle effect arises from the gauge invariance of the metric component $G_{x\phi}$. It implements the conservation of N. We will now argue that in some situations the vacuum of our system has nonzero N which should be cancelled. The low energy degrees of freedom in our cylinder are conformally invariant. Therefore, it is reasonable to assign to each mode in (5.1) the eigenvalues D and $\widetilde{D}$ of the left-moving and right-moving components of the target space energy momentum tensor. We set

$$(D, \widetilde{D}) = \begin{cases} (p_R, 0) & p_R > 0 \\ (0, |p_R|) & p_R < 0 \end{cases} \tag{6.2}$$

The vacuum also carries such values. One way to find them is to follow the computation of (5.6) separately for the left and right-movers

$$\begin{aligned} E &= \frac{1}{2} \sum_{\substack{W=0 \\ p_R>0}} (-1)^F p_R \\ \widetilde{E} &= \frac{1}{2} \sum_{\substack{W=0 \\ p_R<0}} (-1)^F |p_R| \end{aligned} \tag{6.3}$$

with ζ-function regularization. Alternatively, by identifying the boundary conditions of the spacetime fields we can view the ground state of our system as associated with an appropriate twist operator in the target space conformal field theory, whose left and right dimensions are easily determined using standard conformal field theory. (In the target space conformal field theory the parameters E and $\widetilde{E}$ are the conformal dimension minus the central charge over 24.) The upshot of this calculation is that we typically get $E \neq \widetilde{E}$, and therefore the vacuum carries momentum $N = R(E - \widetilde{E})$, which should be cancelled

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by adding an operator with

$$N_{operator} = R(\widetilde{E} - E) \tag{6.4}$$

We now discuss the effect of the gauge field A_ϕ. The vacuum of our system can carry $Spin(8)$ or $Spin(24)$ charges. These arise as follows. The target space fermions in (5.1) with $p_R = 0$ are fermion zero modes. Quantizing them as a Clifford algebra leads to a ground state in a representation of the gauge group. This forces us to turn on an operator in (5.1) in order to form a gauge invariant state. We will discuss this in more detail in the specific examples below.

We conclude that in order to have an invariant state at the weak coupling end we might need to act on the lowest energy state with an operator from (5.1). This operator carries energy $|p_R|$. Since our states evolve along the ϕ direction for "time" V, the effect of this operator insertion is to contribute to the partition function a factor of

$$e^{-V|p_R|} \tag{6.5}$$

The list of operators (5.1) can have many different operators with the required quantum numbers. It is clear from (6.5) that we should take the one with the minimum value of $|p_R|$.

We conclude that we should add another contribution to $\Gamma_{worldsheet}$

$$\Gamma = \Gamma_{worldsheet} - \min |p_R| \tag{6.6}$$

where the minimum is among all the operators with the appropriate charges.

What is the interpretation of this modification of the ϕ-channel vacuum from the point of view of the x-channel? Here we calculate a trace

$$\mathcal{Z} = \mathrm{Tr}\ Oe^{-2\pi RH_x} \tag{6.7}$$

with H_x the Hamiltonian for evolution along x and O an appropriate operator associated with the twisted boundary conditions.

The simplest situation is the case with nonzero W insertion, which occurs in the THO theory. Here the Hilbert space we trace over includes not only the closed string states, but also the long strings and their excitations. The added operator has $p_R = \frac{R}{2} + \frac{b_{long}}{R}$ for some $b_{long} = N$, and therefore it contributes to $\mathcal{Z}$ a factor of

$$\mathcal{Z}_{long} = e^{V\Gamma_{long} + \mathcal{O}(1)} = e^{-V\left(\frac{R}{2} + \frac{1}{R}b_{long}\right) + \mathcal{O}(1)} \tag{6.8}$$

The first term $\frac{RV}{2}$ represents the energy of the long string which arises from its tension. The second term $\frac{V}{R}b_{long}$ represents the effect of the states in the Hilbert space of the long string.

Our other conditions about N and the gauge representation can be interpreted as follows.[6] If we want to view our theory as a free theory in the bulk of space we need to ensure proper boundary conditions at the strong and weak coupling ends of space. We

[6]We thank J. Maldacena for a useful discussion about this point.

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impose that no charge or energy emerges from or leaves the system at the strong coupling end. With Lorentzian signature this means that all scattering processes preserve energy and charge; i.e. the total energy flux and charge which is injected into the system in the past must come back in the future [19]. (This requirement is satisfied with any local boundary conditions at infinity without more degrees of freedom there, but is weaker than such locality.) This means that the integrated energy flux and charge current over all time must vanish

$$\int dx\ T_{x\phi} = \int dx\ J_\phi = 0 \tag{6.9}$$

The naive calculation of the trace and the worldsheet results assume that the left and right-moving modes are independent. Instead, the exact eigenstates of the Hamiltonian are linear combinations of incoming and outgoing modes such that conditions (6.9) are satisfied. The precise linear combination of these modes determines the S-matrix of the theory. In the Euclidean theory with compact x the conditions (6.9) are the same as the invariance conditions of the state that we used above.

Finally, we would like to explain why it is always possible to fix the invariance of the naive lowest energy state with a vertex operator. This follows from the fact that all these compactifications are connected [9]. We can start with a simple circle compactification and continuously (perhaps crossing phase transitions) move to other compactifications. By doing that an invariant ground state undergoes spectral flow to another invariant state. This state can have higher energy than the naive ground state. In that case it is obtained from it by acting with a creation operator.

7. Phase transitions associated with massless fermions

Consider a situation where as we vary R, at some point a massive complex fermion becomes massless and then becomes massive again. The system includes a complex fermion operator Ψ with the Lagrangian and Hamiltonian

$$\begin{aligned} \mathcal{L}_\phi &= i\Psi^\dagger\partial_\phi\Psi + m(R)\Psi^\dagger\Psi \\ H_\phi &= \frac{m(R)}{2}\left(\Psi^\dagger\Psi - \Psi\Psi^\dagger\right) \end{aligned} \tag{7.1}$$

The subscript ϕ denotes the fact that these are the Lagrangian and Hamiltonian in the ϕ-channel. In the examples below the fermion mass is

$$m(R) = \frac{R}{2} - \frac{1}{2R}\,. \tag{7.2}$$

The quantization of such a complex fermion leads to two states $|\pm\rangle$ with energy $\pm\frac{1}{2}m(R)$. For $m > 0$ ($R > 1$) the lowest energy state is $|-\rangle$ and for $m < 0$ ($R < 1$) it is $|+\rangle$. Therefore the ground state energy is $-\frac{1}{2}|m(R)|$. As a check, we will see in the examples below that for $R > 1$ only the operator $\Psi^\dagger$ is present in the list of vertex operators (5.1), while for $R < 1$ only the operator Ψ is associated with a vertex operator. We interpret this to mean that the other operator annihilates the ground state, and therefore it is not included in the list (5.1).

The worldsheet calculations are consistent with this interpretation of the ground state energy. The result, which includes the contribution of all the modes, is

$$\Gamma_{worldsheet} = \frac{1}{2}|m(R)| + f(R) \tag{7.3}$$

with $f(R)$ a smooth function.

This is the correct answer, if we ignore the various gauge symmetries in the problem. We have already seen that possible charges of the vacuum can modify this expression for $R > 1$. In all our examples the complex fermion carries nonzero N and W, and therefore the Lagrangian (7.1) should also include the coupling to the appropriate gauge fields. Therefore, the states $|\pm\rangle$ have opposite charges and hence the lowest energy state for $m > 0$ ($R > 1$) and for $m < 0$ ($R < 1$) have different charges. Consequently, gauge invariance forces us to act on at least one of them, and possibly both, with some vertex operator.

The simplest example is when for $R > 1$ we do not need to act with any operator because the vacuum is invariant. Then, for $R < 1$ we must act with the fermion operator Ψ in order to change the state. This raises the energy of the state by $|m(R)|$ and smooths the transition:

$$\begin{aligned}\Gamma &= \begin{cases} \Gamma_{worldsheet} & R > 1 \\ \Gamma_{worldsheet} - |m| & R < 1 \end{cases} \\ &= \frac{1}{2}m(R) + f(R)\end{aligned} \tag{7.4}$$

Let us discuss this phenomenon in the x-channel and focus on the region where $m(R)$ is small. More precisely, we study the limit

$$m \to 0\ , \qquad V \to \infty\ , \qquad mV = \text{fixed} \tag{7.5}$$

In this limit we can focus on the two states $|\pm\rangle$ of the ϕ-channel picture. For $m > 0$ the closed string partition function $\mathcal{Z}_{closed}$ gives a good approximation to the full answer. It is corrected by the contribution of the long string states. The lowest of them is associated with the fermion $\Psi^\dagger$. Therefore we expect the partition function to be of the form

$$\mathcal{Z}_{worldsheet} = \mathcal{Z}_{closed}\left(1 + c(m)e^{-mV} + \ldots\right) \tag{7.6}$$

where the term $c(m)e^{-Vm}$ represents the contribution of a long string in the system. The exponential factor e^{-Vm} includes the extensive part of the logarithm of the partition function and the prefactor $c(m)$ is a finite size effect. Since this string is fermionic, there is no contribution of the form e^{-nVm} which could have originated from $n > 1$ long strings. The ellipses in (7.6) represent possible contributions which are negligible in the limit (7.5) including the effect of other states which are suppressed by e^{-Vh} for some constant $h \gg |m|$.

Now, let us examine the limit $|mV| \to \infty$. For $m > 0$ the long string contribution vanishes. But for $m < 0$ the second term dominates and $\Gamma_{worldsheet}(R < 1) = \Gamma_{closed} - m(R)$ as in (7.3).

However, this neglects the constraints from gauge invariance. In the simplest situation, where the ϕ-channel vacuum is gauge invariant the second term in (7.6) should not be included and we have $\mathcal{Z} = \mathcal{Z}_{closed}$ for all R. Alternatively, the correct contribution for $R < 1$ is the worldsheet answer $\Gamma_{worldsheet}$ plus the contribution of a long string (7.4). This situation is similar to the more familiar examples including the D0-D8 system [31–34], where a creation of a fundamental string makes the vacuum energy constant when a D0-brane passes through a D8-brane.

Below we will encounter more complicated situations, where the gauge charges force us to act with an operator both above and below the transition point in such a way that the transition is not smoothed out.

8. Phase transitions associated with massless bosons

In this section we will consider a situation where as R is varied a massive complex scalar becomes massless and then massive again.

8.1 ϕ-channel interpretation

In this case the nonanalytic part of Γ is determined by the dynamics of a light complex scalar field Φ. Its effective one-dimensional Lagrangian describes a complex harmonic oscillator

$$\mathcal{L}_\phi = |\partial_\phi \Phi|^2 + m(R)^2 |\Phi|^2 \tag{8.1}$$

where in our examples

$$m(R) = \frac{R}{2} - \frac{1}{2R}\,. \tag{8.2}$$

The boundary conditions at $\phi = -V, 0$ are summarized by (boundary) states $|s_{V,0}\rangle$ in the ϕ-channel Hilbert space. In this language the functional integral is easily evaluated as a sum over intermediate states. Gauge invariance forces us to consider boundary states which are invariant under the $U(1)$ gauge symmetry $\Phi \to e^{i\alpha}\Phi$. We assume, for simplicity, that our vacuum is $U(1)$ invariant (more complicated situations will be discussed below). Then, only the $U(1)$ invariant states $|n\rangle$ with energy $(2n+1)|m|$ contribute

$$\begin{aligned}\mathcal{Z}_{oscillator} &= \langle s_0|e^{-VH_\phi}|s_V\rangle = \sum_{n=0}^{\infty} c_n(m) e^{-(2n+1)|m|V} \\ c_n(m) &= \langle s_0|n\rangle\langle n|s_V\rangle\end{aligned} \tag{8.3}$$

where H_ϕ is the ϕ-channel Hamiltonian of the Lagrangian (8.1). The V dependence of each term $e^{-(2n+1)|m|V}$ is independent of the boundary conditions and depends only on the energy of the state $|n\rangle$. However, the prefactors c_n are "finite volume corrections" which depend on the boundary conditions.

Here are three simple illustrative examples with $|s_{V,0}\rangle$ being Φ eigenstates with zero eigenvalues or P_Φ eigenstates with zero eigenvalue

$$\langle \Phi = 0|e^{-VH_\phi}|\Phi = 0\rangle = \frac{m}{2\pi \sinh(mV)} = \frac{|m|}{\pi} \sum_{n=0}^{\infty} e^{-(2n+1)|m|V}$$

$$\langle P_\Phi = 0|e^{-VH_\phi}|P_\Phi = 0\rangle = \frac{1}{2\pi m \sinh(mV)} = \frac{1}{\pi|m|}\sum_{n=0}^{\infty} e^{-(2n+1)|m|V}$$
$$\langle P_\Phi = 0|e^{-VH_\phi}|\Phi = 0\rangle = \frac{1}{2\pi \cosh(mV)} = \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n e^{-(2n+1)|m|V} \tag{8.4}$$

As we will see in the examples below, our string problem has T-duality symmetry which maps $m \to -m$. Furthermore, in the HE and HO theories at the selfdual point $m = 0$ the T-duality transformation is part of an enhanced non-Abelian target space symmetry (gauge symmetry of the ϕ-channel problem). Therefore, it must be an exact symmetry of the system [35], and hence the states $|s_{V,0}\rangle$ should be independent of the sign of m. Hence, the coefficients $c_n(m)$ and the partition function (8.3) should also be independent of the sign of m. This is the case in the examples (8.4).

If $c_0(m) \neq 0$, the large V limit of $\mathcal{Z}_{oscillator}$ is dominated by the ground state

$$\lim_{V\to\infty} \mathcal{Z}_{oscillator} = c_0(m)e^{-|m|V} . \tag{8.5}$$

It is nonanalytic in m around $m = 0$. However, the finite V expressions are analytic in m around $m = 0$. Using (8.3) we find for normalizable states

$$\lim_{m\to 0} \mathcal{Z}_{oscillator} = \sum_{n=0}^{\infty} c_n(m) = \langle s_0|s_V\rangle \tag{8.6}$$

which is finite. Even for delta-function normalizable states, as in the examples (8.4), it is easy to check that $\lim_{m\to 0}\mathcal{Z}_{oscillator}$ is either finite or has a double pole. Therefore, for finite V we can analytically continue the expressions from positive to negative m.

Now we embed this harmonic oscillator in our string problem. We assume that both $|s_0\rangle$ and $|s_V\rangle$ have nonzero overlap with the harmonic oscillator vacuum $|0\rangle$ so that c_0 in (8.3) is nonzero. Including all the other modes we conclude that the infinite V worldsheet expression is nonanalytic . It is given by

$$\Gamma_{worldsheet} = -|m(R)| + f(R) \tag{8.7}$$

with $f(R)$ a smooth function. This will turn out to be consistent with explicit worldsheet calculations and T-duality.

8.2 x-channel interpretation — puzzling thermodynamics

So far we discussed the transition from the point of view of the ϕ-channel. We now discuss it in the x-channel, where ϕ is viewed as space and x as time. As we will see in the examples below, this transition happens whenever our system is compactified on a thermal circle; i.e. twisted by $(-1)^F$ with F the spacetime fermion number. Usually this means that the x-channel system is at finite temperature with the partition function

$$\mathcal{Z} = \mathrm{Tr}\, e^{-2\pi R H_x} \tag{8.8}$$

where H_x is the x-channel Hamiltonian.

The expression (8.8) has two immediate standard consequences. First, the partition function can be written as a sum of positive numbers, and therefore $\mathcal{Z} > 0$. Second, the expectation value of positive quantities like $\left\langle (E - \langle E \rangle)^2 \right\rangle$ should be positive, and therefore the specific heat must be positive

$$c \sim R^2 \partial_R^2 \Gamma > 0 \tag{8.9}$$

For simplicity we focus on situations where the ϕ-channel ground state is invariant and no long strings are needed. This will be the case in the HE and HO theories. The extension to more complicated theories like THO is straightforward.

We start the discussion for $R > 1$ where $m > 0$. Here Γ_{closed} which is computed as $\mathrm{Tr}\, e^{-2\pi R H_x}$ with the closed string spectrum agrees with $\Gamma_{worldsheet}$. However, an important subtlety has to be stressed. In addition to the closed strings, the Hilbert space includes also the long folded strings of figure 2. As we said above, such states have large energy proportional to the large volume V, and therefore for sufficiently large R (small temperature) they do not contribute to the large V limit of the partition function. Because of the degrees of freedom on them they also have large entropy proportional to $\frac{V}{R^2}$. This has no effect for $R > 1$. However, for $R = 1$ the gas of such long strings can contribute to the partition function. This explains why the answer Γ_{closed} is correct for $R > 1$ but is wrong for $R \leq 1$. It simply misses the contribution of these states in the Hilbert space.

Let us analyze these folded long strings in more detail. As we said above, the energy of such a folded string is proportional to the length $V + \phi_0$ and diverges in the thermodynamic limit $V \to \infty$. This dependence on ϕ_0 leads to an attractive force proportional to R toward the weak coupling end. Hence such a folded string is unstable and retracts back to the cutoff at the weak coupling end [18]. However, unlike the examples studied in [18], in our case, such a folded string has oscillators living on it, and they lead to entropy. This produces a force proportional to $\frac{1}{R}$ pushing $\phi_0 \to +\infty$. For $R > 1$ the attraction to $\phi_0 \to -\infty$ is the dominant effect, but for $R = 1$ these two forces are exactly balanced. It seems that for $R < 1$ these strings are attracted to the strong coupling end and our system will have a condensate of such long strings.

This long string condensation picture is similar to the picture of the Hagedorn transition in higher dimensions. There the transition is associated with the condensation of long closed strings [36, 37]. In our case, the closed strings do not have the necessary entropy to lead to Hagedorn behavior. Instead, a closed string which is stretched all the way to infinity is the folded string. It has the necessary entropy to lead to a transition.

We would like to study the region near the transition point at large but finite V. Therefore, we study the limit (7.5)

$$m \to 0\ , \qquad V \to \infty\ , \qquad mV = \text{fixed} \tag{8.10}$$

Our discussion of the ϕ-channel picture suggests that for finite V the m dependence can be analytically continued to $m < 0$ ($R < 1$). Before we see where this analytic continuation leads us, we examine the partition function for $m > 0$ ($R > 1$) in the limit (8.10), making the intuitive picture above more quantitative. The ϕ-channel expression (8.3) shows that for $m > 0$ the partition function is

$$\mathcal{Z} \approx \mathcal{Z}_{closed} \left(1 + \sum_{n=1}^{\infty} \frac{c_n}{c_0} e^{-2nmV} \right) \tag{8.11}$$

where we have identified the first term as $\mathcal{Z}_{closed}$. In the ϕ-channel picture the various terms in the sum in (8.11) are associated with the excited states of the harmonic oscillator. We now interpret them in the x-channel as representing the contributions of different number of pairs of long strings. Each pair has a partition function

$$\mathcal{Z}_{pair} \sim e^{-2mV} \tag{8.12}$$

where the factor of 2 is because of the string and the anti-string. As we have already remarked, for $m(R) = \frac{R}{2} - \frac{1}{2R}$ the first term $\frac{R}{2}$ is due to the energy of the long string and the second term $\frac{1}{2R}$ represents its entropy.

In order for this x-channel interpretation to be sensible, we need $\frac{c_n}{c_0}$ to be positive real numbers for all n. Clearly, the generic boundary state in (8.3) does not satisfy this requirement (see, e.g. the examples (8.4)), but boundary conditions at $\phi = -V, 0$ which are local in x satisfy it. Let us examine whether such local boundary conditions are possible.

We have discussed the expansion (8.11) for $m > 0$. Since we are studying the finite volume system, it is reasonable that it converges for most m and defines an analytic function of m. This expectation is supported by the ϕ-channel expression (8.3) which defines such an analytic function (see e.g. the examples (8.4)). Furthermore, as we mentioned above, our system should be invariant under the T-duality transformation which acts as $m \to -m$. This leads to (8.5) and (8.7)

$$\Gamma = \lim_{V\to\infty} \frac{1}{V} \log(\mathcal{Z}) = \begin{cases} \Gamma_{closed} & m > 0 \\ \Gamma_{closed} - 2|m| & m < 0 \end{cases} \tag{8.13}$$

However, (8.13) exhibit a problem – the system has a first order phase transition at $m = 0$ ($R = 1$) whose latent heat is negative. It is easy to see that any smooth finite V expression which asymptotes to (8.13) as $V \to \infty$ violates (8.9); i.e. it has negative specific heat. Since (8.9) follows from the positivity of the weights in the trace over the Hilbert space, our answer (8.13) cannot arise in any thermodynamical system.

We conclude that for infinite V and negative m there is a conflict between the ϕ-channel and the x-channel pictures. For finite V this conflict arises for all m but for positive m the problem is exponentially small in V. This conflict can be traced back to the contribution of the ground state energy of the harmonic oscillator in the ϕ-channel being $|m|$ and the long string contribution $\Gamma_{long} = -|m| = -\left|\frac{R}{2} - \frac{1}{2R}\right|$ and $\Gamma_{pair} = 2\Gamma_{long} = -\left|R - \frac{1}{R}\right|$ all with absolute values. On the other hand, the x-channel picture has, as in (8.12) $\Gamma_{pair} = -2m = -\left(R - \frac{1}{R}\right)$ without absolute value (recall the interpretation of the first term R as the associated with the energy and the second term $\frac{1}{R}$ as associated with the entropy).

We suggest that the boundary states $|s_{V,0}\rangle$ which are invariant under $m \to -m$ and in particular the ground state $|0\rangle$ are associated with boundary conditions which are nonlocal in x. Such nonlocal boundary conditions do not satisfy the thermodynamical positivity of

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the x-channel picture. A simple example is the third inner product in (8.4). The expansion in the right hand side has both positive and negative quantities and cannot be given an interpretation as $\mathrm{Tr}\, e^{-2\pi R H_x}$ in the x-channel.

We conclude that for negative m the ϕ-channel quantization is nonlocal in x and cannot be interpreted as thermodynamics in the x-channel. Instead, it has another interpretation with ϕ as space describing the thermodynamics of the T-dual system, whose radius is $\frac{1}{R}$ and is associated with local evolution along the T-dual circle. For $R<1$ the expressions $\Gamma_{long} = -|m| = -\left|\frac{R}{2} - \frac{1}{2R}\right|$ and $\Gamma_{pair} = 2\Gamma_{long} = -\left|R - \frac{1}{R}\right|$ describe not a long string and a long pair but their T-dual – a long string and a long pair in the T-dual theory.

To summarize, the ϕ-channel picture leads to a sensible expression for the partition function of the Euclidean circle problem for all R. This expression agrees with thermodynamics in the x channel only for $R>1$. For $R<1$ it agrees with thermodynamics of the T-dual theory on a circle of radius $\frac{1}{R}$. However, these two thermodynamics interpretations cannot be continued to higher temperatures. We suggest that the sensible answers derived from the ϕ-channel are associated with nonlocal boundary conditions in x and this nonlocality prevents us from having thermodynamics at higher temperatures. We will return to this point in the conclusions

9. Untwisted circle

We now turn to various examples making our general discussion above more concrete.

The simplest compactification without a twist of HO and HE has the vertex operators [9]

$$\begin{aligned} T_V &= e^{-\varphi}\overline{\mathcal{O}}_V\left(\frac{1}{2}+nw\right)V_{n,w} \\ \Psi_S &= e^{-\frac{1}{2}\varphi+i\frac{1}{2}H}\overline{\mathcal{O}}_S\left(\frac{1}{2}+nw\right)V_{n,w}\ , \qquad p_R\geq 0 \\ \widetilde{\Psi}_C &= e^{-\frac{1}{2}\varphi-i\frac{1}{2}H}\overline{\mathcal{O}}_C\left(\frac{1}{2}+nw\right)V_{n,w}\ , \qquad p_R\leq 0 \end{aligned} \tag{9.1}$$

In the THO theory we have [10]

$$\begin{aligned} T_C &= e^{-\varphi}\overline{\mathcal{O}}_C\left(\frac{1}{2}+nw\right)V_{n,w} \\ \Psi_S &= e^{-\frac{1}{2}\varphi+i\frac{1}{2}H}\overline{\mathcal{O}}_S\left(\frac{1}{2}+nw\right)V_{n,w}\ , \qquad p_R\geq 0 \\ \widetilde{\Psi}_V &= e^{-\frac{1}{2}\varphi-i\frac{1}{2}H}\overline{\mathcal{O}}_V\left(\frac{1}{2}+nw\right)V_{n,w}\ , \qquad p_R\leq 0\,. \end{aligned} \tag{9.2}$$

These three compactifications are invariant under the T-duality transformation $R\to\frac{2}{R}$, with enhanced $SU(2)$ symmetry at the selfdual point $R=\sqrt{2}$.

The field theory calculation of Γ_{closed}, based on the spectrum of the theory in noncompact spacetime is straightforward [9, 10]

$$\Gamma_{closed} = \begin{cases} R + \frac{2}{R} & HO \\ 0 & HE \\ -\frac{R}{2} - \frac{1}{R} & THO \end{cases} \tag{9.3}$$

The analysis of [9] shows that in these theories the worldsheet calculation leads to the same answer for all R

$$\Gamma_{worldsheet}(R) = \Gamma_{closed}(R) \tag{9.4}$$

Let us examine whether the ϕ-channel ground state has to be modified. First, in HO and HE we must have $W = 0$, and in THO we should add $W = 1$. Second, it is easy to check using (6.3) (6.4) that in HO and HE we need $N = 0$, while in THO we need $N = 1$.

Next we check the gauge charges of the vacuum. In the HO theory there are no fermion zero modes, and therefore there is no subtlety in the quantization.

The HE theory has two fermion zero modes $\Psi_S(N = W = 0)$, whose quantization leads to $\mathbf{8}_V \oplus \mathbf{8}_C$, and $\widetilde{\Psi}_C(N = W = 0)$, whose quantization leads to $\mathbf{8}_V \oplus \mathbf{8}_S$. There is a unique gauge invariant state in $\mathbf{8}_V \otimes \mathbf{8}_V$, and therefore no modification of the spectrum is needed.

The THO theory has one fermion zero mode $\widetilde{\Psi}_V(N = W = 0)$ whose quantization leads to $\mathbf{2^{11}}_S \oplus \mathbf{2^{11}}_C$. The lack of gauge invariance of this representation is cancelled by acting on the lowest energy state with a vertex operator. Combining this information with the fact that the operator should have $N = W = 1$, we find that this operator is $T_C(N = W = 1)$ or $\Psi_S(N = W = 1)$. We interpret these operators as associated with the long string that must be added to the system.

We conclude that

$$\Gamma = \begin{cases} R + \frac{2}{R} & HO \\ 0 & HE \\ -\frac{R}{2} - \frac{1}{R} - \left(\frac{R}{2} + \frac{1}{R}\right) = -R - \frac{2}{R} & THO \end{cases} \tag{9.5}$$

As a check, note that this answer is invariant under the T-duality transformation $R \to \frac{2}{R}$.

Since as we vary R no mode becomes massless and correspondingly Γ_{long} is always positive, there cannot be a phase transition in this system, and therefore the expressions (9.5) are correct for all R.

10. Twisted circles

10.1 Twist by $(-1)^{f_L}$

We now consider compactifications which are twisted by $(-1)^{f_L}$ with f_L the worldsheet fermion number of the left-movers. Such twists are possible only if the tachyon deformation (2.4) vanishes. Also, this twist breaks the spacetime parity symmetry. One way to see that

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is to note that $(-1)^{f_L}$ acts differently on the representations with C and S, which are exchanged by spacetime parity.

The spectrum of vertex operators in HE and HO is [9]

$$\begin{aligned}
T_V &= e^{-\varphi}\overline{\mathcal{O}}_V\left(\frac{1}{2}+(2n+1)w\right)V_{n+\frac{1}{2},2w} \\
T_C &= e^{-\varphi}\overline{\mathcal{O}}_C\left(\frac{1}{2}+n(2w+1)\right)V_{n,2w+1} \\
\Psi_S &= e^{-\varphi/2+iH/2}\overline{\mathcal{O}}_S\left(\frac{1}{2}+2nw\right)V_{n,2w}\ , && p_R\geq 0 \\
\Psi_0 &= e^{-\varphi/2+iH/2}\overline{\mathcal{O}}_0\left(\frac{1}{2}+\left(n+\frac{1}{2}\right)(2w+1)\right)V_{n+\frac{1}{2},2w+1}\ , && p_R\geq 0 \\
\widetilde{\Psi}_C &= e^{-\varphi/2-iH/2}\overline{\mathcal{O}}_C\left(\frac{1}{2}+(2n+1)w\right)V_{n+\frac{1}{2},2w}\ , && p_R\leq 0 \\
\widetilde{\Psi}_V &= e^{-\varphi/2-iH/2}\overline{\mathcal{O}}_V\left(\frac{1}{2}+n(2w+1)\right)V_{n,2w+1}\ , && p_R\leq 0
\end{aligned} \tag{10.1}$$

and in THO it is [10]

$$\begin{aligned}
T_V &= e^{-\varphi}\overline{\mathcal{O}}_V\left(\frac{1}{2}+n(2w+1)\right)V_{n,2w+1} \\
T_C &= e^{-\varphi}\overline{\mathcal{O}}_C\left(\frac{1}{2}+(2n+1)w\right)V_{n+\frac{1}{2},2w} \\
\Psi_S &= e^{-\varphi/2+iH/2}\overline{\mathcal{O}}_S\left(\frac{1}{2}+2nw\right)V_{n,2w}\ , && p_R\geq 0 \\
\Psi_0 &= e^{-\varphi/2+iH/2}\overline{\mathcal{O}}_0\left(\frac{1}{2}+\left(n+\frac{1}{2}\right)(2w+1)\right)V_{n+\frac{1}{2},2w+1}\ , && p_R\geq 0 \\
\widetilde{\Psi}_C &= e^{-\varphi/2-iH/2}\overline{\mathcal{O}}_C\left(\frac{1}{2}+n(2w+1)\right)V_{n,2w+1}\ , && p_R\leq 0 \\
\widetilde{\Psi}_V &= e^{-\varphi/2-iH/2}\overline{\mathcal{O}}_V\left(\frac{1}{2}+(2n+1)w\right)V_{n+\frac{1}{2},2w}\ , && p_R\leq 0
\end{aligned} \tag{10.2}$$

HE is selfdual (with $V\leftrightarrow C$) under $R\to\frac{1}{R}$ with an enhanced $Spin(10)\times E_8$ symmetry at the selfdual point $R=1$. The HO theory compactified on a circle of radius R is dual to the THO on a circle of radius $\frac{1}{R}$. At the point $R=1$ these theories include the worldsheet current $e^{-\varphi/2+iH/2}V_{\frac{1}{2},1}=e^{-\varphi/2+iH/2+ix}$, which leads to an anticommuting target space (super)symmetry. At this point these theories are special cases of the noncritical superstring construction of [38].

The field theory calculation of the closed strings leads to [9, 10]

$$\Gamma_{closed}=\begin{cases} -\frac{1}{2R} & HE \\ R-\frac{1}{R} & HO \\ -\frac{R}{2}+\frac{1}{2R} & THO \end{cases} \tag{10.3}$$

and the worldsheet answers are [9, 10]

$$\Gamma_{worldsheet}(R>1)=\Gamma_{closed}(R)$$

$$\Gamma_{worldsheet}(R<1) = \Gamma_{closed}(R) - \left(\frac{R}{2} - \frac{1}{2R}\right) = \begin{cases} -\frac{R}{2} & HE \\ \frac{R}{2} - \frac{1}{2R} & HO \\ -R + \frac{1}{R} & THO \end{cases} \tag{10.4}$$

where we see the effect of the massless fermion at $R = 1$.

We now consider the necessary modifications due to gauge invariance, and start with the situation for $R > 1$. There is no need to change the HO theory. The THO theory needs an insertion of a long string ($W = 1$) with $N = -\frac{1}{2}$; i.e. we act on the lowest energy state with $\Psi_0(N = -\frac{1}{2}, W = 1)$.

The HE theory is more interesting. Here we need to add an operator with $W = 0$ and $N = -\frac{1}{2}$. This is our only example where the added operator has $W = 0$, and therefore the modification of the theory cannot be interpreted as adding a long string. For generic R we have the fermion zero mode $\Psi_S(N = W = 0)$, whose quantization leads to a ground state in $\mathbf{8}_V \oplus \mathbf{8}_C$. Therefore, we have to act on the ϕ-channel lowest energy state with $T_V(N = -\frac{1}{2}, W = 0)$ or $\widetilde{\Psi}_C(N = -\frac{1}{2}, W = 0)$. The fact that we can achieve gauge invariance with zero W is consistent with the fact that the noncompact theory does not have long strings.

We conclude that for $R > 1$

$$\Gamma(R>1) = \begin{cases} -\frac{1}{2R} - \frac{1}{2R} = -\frac{1}{R} & HE \\ R - \frac{1}{R} & HO \\ -\frac{R}{2} + \frac{1}{2R} - \left(\frac{R}{2} - \frac{1}{2R}\right) = -R + \frac{1}{R} & THO \end{cases} \tag{10.5}$$

Let us now move to $R < 1$. In all these theories a complex fermion becomes massless at $R = 1$. Therefore, as we explained above, the change in the quantum numbers of the lowest energy state forces us to act with $\Psi_0(N = \frac{1}{2}, W = -1)$. In the HO theory this adds a long string and in THO this removes the added long string. In the HE theory we act on the naive ground state with the lowest dimension operator with the quantum numbers of $\Psi_0(N = \frac{1}{2}, W = -1)$ $T_V(N = -\frac{1}{2}, W = 0)$ which is $\widetilde{\Psi}_V(N = 0, W = -1)$, or with the lowest dimension operator with the quantum numbers of $\Psi_0(N = \frac{1}{2}, W = -1)$ $\widetilde{\Psi}_C(N = -\frac{1}{2}, W = 0)$ which is $T_C(N = 0, W = -1)$. This adds an anti-long string ($W = -1$), but since it has $N = 0$, its T-dual version does not involve a long string.

We conclude that

$$\Gamma(R<1) = \Gamma_{worldsheet}(R<1) - \begin{cases} \frac{R}{2} & HE \\ -\frac{R}{2} + \frac{1}{2R} & HO \\ 0 & THO \end{cases} = \begin{cases} -R & HE \\ R - \frac{1}{R} & HO \\ -R + \frac{1}{R} & THO \end{cases} \tag{10.6}$$

Comparing with (10.5) we see that HE is indeed selfdual and HO and THO are T-dual to each other under $R \to \frac{1}{R}$. Evidently, the phase transition in HO and THO at $R = 1$ is smoothed out, but the HE theory still has a phase transition there.

10.2 Twist by $(-1)^{f_L+F}$

Another possibility is to twist the circle by $(-1)^{f_L+F}$, where F is the spacetime fermion number. In the HO and HE theory this twisted compactification is the same as the one

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with $(-1)^{f_L}$ (section 10.1). The vertex operators of THO are [10]

$$\begin{aligned}
T_C &= e^{-\varphi}\overline{\mathcal{O}}_C\left(\frac{1}{2}+(2n+1)w\right)V_{n+\frac{1}{2},2w} \\
T_S &= e^{-\varphi}\overline{\mathcal{O}}_S\left(\frac{1}{2}+n(2w+1)\right)V_{n,2w+1} \\
\Psi_S &= e^{-\frac{1}{2}\varphi+i\frac{1}{2}H}\overline{\mathcal{O}}_S\left(\frac{1}{2}+(2n+1)w\right)V_{n+\frac{1}{2},2w}\ , \qquad\qquad p_R\geq 0 \\
\Psi_C &= e^{-\frac{1}{2}\varphi+i\frac{1}{2}H}\overline{\mathcal{O}}_C\left(\frac{1}{2}+n(2w+1)\right)V_{n,2w+1}\ , \qquad\qquad p_R\geq 0 \\
\widetilde{\Psi}_0 &= e^{-\frac{1}{2}\varphi-i\frac{1}{2}H}\overline{\mathcal{O}}_0\left(\frac{1}{2}+\left(n+\frac{1}{2}\right)(2w+1)\right)V_{n+\frac{1}{2},2w+1}\ , \quad p_R\leq 0 \\
\widetilde{\Psi}_V &= e^{-\frac{1}{2}\varphi-i\frac{1}{2}H}\overline{\mathcal{O}}_V\left(\frac{1}{2}+2nw\right)V_{n,2w}\ , \qquad\qquad p_R\leq 0
\end{aligned} \tag{10.7}$$

This compactification is selfdual (with $S\leftrightarrow C$) under $R\to\frac{1}{R}$ with enhanced $Spin(26)$ symmetry at the selfdual point $R=1$. At this point the theory includes the worldsheet current $e^{-\varphi/2-iH/2}V_{-\frac{1}{2},-1}=e^{-\varphi/2-iH/2-ix}$, which leads to an anticommuting target space (super)symmetry. Again, this is a special cases of the noncritical superstring construction of [38].

Here [10]

$$\begin{aligned}
\Gamma_{closed} &= -\frac{R}{2}-\frac{1}{R} \\
\Gamma_{worldsheet}(R>1) &= \Gamma_{closed}(R>1) \\
\Gamma_{worldsheet}(R<1) &= \Gamma_{closed}(R<1)-\left(\frac{R}{2}-\frac{1}{2R}\right)=-R-\frac{1}{2R}
\end{aligned} \tag{10.8}$$

where again we see the effect of the massless fermion at $R=1$.

For $R>1$ we have to act on the lowest energy state with an operator with $W=1$ and $N=1$. The $Spin(24)$ quantum numbers are determined by noting that $\widetilde{\Psi}_V(N=W=0)$ is the only fermion zero mode. Its quantization leads to $\mathbf{2^{11}}_S\oplus\mathbf{2^{11}}_C$. Therefore, we must act with $T_S(N=W=1)$ or $\Psi_C(N=W=1)$ and

$$\Gamma(R>1)=-\frac{R}{2}-\frac{1}{R}-\left(\frac{R}{2}+\frac{1}{R}\right)=-R-\frac{2}{R} \tag{10.9}$$

For $R<1$ we must also act with $\widetilde{\Psi}_0(N=-\frac{1}{2},W=1)$. Therefore, we act on the naive ground state with $\widetilde{\Psi}_0(N=-\frac{1}{2},W=1)\ T_S(N=W=1)\sim\Psi_S(N=\frac{1}{2},W=2)$ or $\widetilde{\Psi}_0(N=-\frac{1}{2},W=1)\ \Psi_C(N=W=1)\sim T_C(N=\frac{1}{2},W=2)$. This can be interpreted as having two long strings in the target space. The fact that $N=\frac{1}{2}$ shows that in the T-dual picture there is only one long string. Hence

$$\Gamma(R<1)=\Gamma_{worldsheet}(R<1)-\left(\frac{1}{2R}+R\right)=-2R-\frac{1}{R} \tag{10.10}$$

Comparing with (10.9) we see that the final answer is selfdual.

11. Thermal circle

Here we compactify our theories on a thermal circle; i.e. we twist by $(-1)^F$ with F the spacetime fermion number.

The spectrum of operators of the HO and HE theories is [9]

$$\begin{aligned}
T_V &= e^{-\varphi}\overline{\mathcal{O}}_V\left(\frac{1}{2}+2nw\right)V_{n,2w}\\
T_0 &= e^{-\varphi}\overline{\mathcal{O}}_0\left(\frac{1}{2}+\left(n+\frac{1}{2}\right)(2w+1)\right)V_{n+\frac{1}{2},2w+1}\\
\Psi_S &= e^{-\frac{1}{2}\varphi+i\frac{1}{2}H}\overline{\mathcal{O}}_S\left(\frac{1}{2}+(2n+1)w\right)V_{n+\frac{1}{2},2w}\,, && p_R\geq 0\\
\Psi_C &= e^{-\frac{1}{2}\varphi+i\frac{1}{2}H}\overline{\mathcal{O}}_C\left(\frac{1}{2}+n(2w+1)\right)V_{n,2w+1}\,, && p_R\geq 0\\
\widetilde{\Psi}_S &= e^{-\frac{1}{2}\varphi-i\frac{1}{2}H}\overline{\mathcal{O}}_S\left(\frac{1}{2}+n(2w+1)\right)V_{n,2w+1}\,, && p_R\leq 0\\
\widetilde{\Psi}_C &= e^{-\frac{1}{2}\varphi-i\frac{1}{2}H}\overline{\mathcal{O}}_C\left(\frac{1}{2}+(2n+1)w\right)V_{n+\frac{1}{2},2w}\,, && p_R\leq 0
\end{aligned} \tag{11.1}$$

These compactifications are T-duality invariant under $R\to\frac{1}{R}$. This duality is guaranteed by an enhanced symmetry at the selfdual point $Spin(24)\times U(1)\to Spin(26)$ and $Spin(8)\times E_8\times U(1)\to Spin(10)\times E_8$.

It is interesting to compare this compactification with the untwisted circle compactification (9.1) of the HO theory. The closed string spectrum of HO in $\mathbb{R}^2$ includes only bosons, and therefore it is not affected by the thermal boundary conditions. Nevertheless, the thermal circle (11.1) differs from the untwisted circle (9.1). One way to understand it is to note that the two different twists around the circle act differently on the long strings. This difference manifests itself in the different spectrum of operators with nonzero W in these two compactifications.

The spectrum of closed string vertex operators in the thermal THO theory is [10]

$$\begin{aligned}
T_C &= e^{-\varphi}\overline{\mathcal{O}}_C\left(\frac{1}{2}+2nw\right)V_{n,2w}\\
T_0 &= e^{-\varphi}\overline{\mathcal{O}}_0\left(\frac{1}{2}+\left(n+\frac{1}{2}\right)(2w+1)\right)V_{n+\frac{1}{2},2w+1}\\
\Psi_S &= e^{-\frac{1}{2}\varphi+i\frac{1}{2}H}\overline{\mathcal{O}}_S\left(\frac{1}{2}+(2n+1)w\right)V_{n+\frac{1}{2},2w}\,, && p_R\geq 0\\
\Psi_V &= e^{-\frac{1}{2}\varphi+i\frac{1}{2}H}\overline{\mathcal{O}}_V\left(\frac{1}{2}+n(2w+1)\right)V_{n,2w+1}\,, && p_R\geq 0\\
\widetilde{\Psi}_S &= e^{-\frac{1}{2}\varphi-i\frac{1}{2}H}\overline{\mathcal{O}}_S\left(\frac{1}{2}+n(2w+1)\right)V_{n,2w+1}\,, && p_R\leq 0\\
\widetilde{\Psi}_V &= e^{-\frac{1}{2}\varphi-i\frac{1}{2}H}\overline{\mathcal{O}}_V\left(\frac{1}{2}+(2n+1)w\right)V_{n+\frac{1}{2},2w}\,, && p_R\leq 0\,.
\end{aligned} \tag{11.2}$$

This spectrum is invariant under $R\to\frac{1}{R}$ up to spacetime symmetry. This T-duality is not guaranteed by an enhanced non-Abelian symmetry at the selfdual point.

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The closed strings lead to [9, 10]

$$
\begin{aligned}
\log\left(\mathcal{Z}_{closed}\right) = & \quad V\Gamma_{closed} + \mathcal{O}(1) \\
\Gamma_{closed} = & \begin{cases} \frac{1}{R} & HE \\ R + \frac{2}{R} & HO \\ -\frac{R}{2} + \frac{1}{2R} & THO \end{cases}
\end{aligned}
\tag{11.3}
$$

and the worldsheet answers are [9, 10]

$$
\begin{aligned}
\Gamma_{worldsheet}(R>1) &= \Gamma_{closed}(R>1) \\
\Gamma_{worldsheet}(R<1) &= \Gamma_{closed}(R<1) + \left(R - \frac{1}{R}\right) = \begin{cases} R & HE \\ 2R + \frac{1}{R} & HO \\ \frac{R}{2} - \frac{1}{2R} & THO \end{cases}
\end{aligned}
\tag{11.4}
$$

where we see the effect of the massless boson at $R = 1$.

Repeating the analysis of the charges of the lowest energy state in the ϕ-channel, we find that only in the THO theory the ground state needs a modification, and that is achieved by acting on it with $T_0(N = -\frac{1}{2}, W = 1)$. Therefore, $\Gamma = \Gamma_{worldsheet} - \left|\frac{R}{2} - \frac{1}{2R}\right|$. We conclude that

$$
\begin{aligned}
\Gamma(R>1) &= \begin{cases} \frac{1}{R} & HE \\ R + \frac{2}{R} & HO \\ -\frac{R}{2} + \frac{1}{2R} - \left(\frac{R}{2} - \frac{1}{2R}\right) = -R + \frac{1}{R} & THO \end{cases} \\
\Gamma(R<1) &= \begin{cases} \Gamma_{worldsheet}(R<1) = R & HE \\ \Gamma_{worldsheet}(R<1) = 2R + \frac{1}{R} & HO \\ \Gamma_{worldsheet}(R<1) - \left|\frac{R}{2} - \frac{1}{2R}\right| = R - \frac{1}{R} & THO \end{cases}
\end{aligned}
\tag{11.5}
$$

As expected, these results T-dual. Note that the added string in THO with $W = 1$ and $N = -\frac{1}{2}$ becomes after T-duality a string with $W = -1$ and $N = \frac{1}{2}$. Its opposite orientation fits with the fact that the T-dual theory is the parity image of THO.

12. Conclusions

We have studied the two-dimensional heterotic string uncovering a number of new phenomena:

1. The THO theory has anomalies which are cancelled by adding a single infinitely long string stretched across the space.

2. The spectrum of all our theories includes pairs of a long string and a long anti-string, or equivalently, the folded string of figure 2. These infinite energy excitations are important in the study of compactifications.

3. After compactification of the x direction we find a generalization of the need to add a long string. In some of our examples a careful analysis of the gauge symmetries of the compactified theory shows that the lowest energy state in the ϕ-channel is not

invariant. This lack of gauge invariance can be fixed by acting on it with a creation operator. This has the effect of changing the worldsheet answer. The latter always describes the lowest energy state whether it is invariant or not.

4. As we vary the parameters of the compactifications including the radius R and Wilson lines various phase transitions can take place. The simplest kind of phase transition is associated with massless fermions. Since the fermions are charged, the lowest energy states in the ϕ-channel in the two sides of the transition have different quantum numbers. Therefore, at least on one side of the transition (and possibly on both sides) we need to act on the lowest energy state with a vertex operator. This changes the transition and sometimes smoothes it out.

5. Our most dramatic conclusions are associated with phase transitions associated with massless bosons. These arise, among other places, in thermal compactifications. It seems that we have a conflict between T-duality, locality in ϕ, locality in x and locality in the dual of x. One possibility is to follow the quantization in the ϕ-channel and preserve T-duality. Then we have standard thermodynamics in the x-channel for $R > 1$ (low temperatures) and standard thermodynamics in the dual of x for $R < 1$ (which is again low temperatures, in the T-dual picture). But the x-channel thermodynamics cannot be continued to higher temperatures.

We would like to make a few more comments about the last point regarding the thermal circle.

Let us compare our thermal system with the thermal ten-dimensional heterotic string. In both cases we find a thermal phase transition due to the winding mode with $N = -\frac{1}{2}$, $W = 1$ and its complex conjugate with $N = \frac{1}{2}$, $W = -1$. The corresponding worldsheet vertex operator is relevant for $\sqrt{2} - 1 < R < 1 + \sqrt{2}$. In the critical string it is massless at $R = \sqrt{2} \pm 1$, and it is tachyonic between these two points. This is the standard Hagedorn transition. In the two-dimensional heterotic string the vertex operator is still relevant for $\sqrt{2} - 1 < R < 1 + \sqrt{2}$, but it is never tachyonic. It is massless only at the selfdual point $R = 1$. Therefore, we expect the transition in two dimensions to be milder than in ten dimensions.

We assume that these heterotic theories have dual descriptions in terms of boundary theories. What do our results imply about these holographic descriptions? The standard AdS/CFT [39], and the $c \leq 1$ and $\widehat{c} \leq 1$ theories have a dual description in terms of a local quantum field theory. Even though our heterotic theories have a large dilaton slope and only a finite number of massless particles in the two-dimensional bulk, they are similar to higher dimensional linear dilaton theories like the little string theory [40][41][42]–[43]. These holographic theories do not appear to be local [41, 44, 45], and they have peculiar finite temperature behavior [45].

One might ask wether our analysis, which focuses only on the bulk of the target space and ignores the strong coupling region, can be misleading. In particular, can the system have instabilities which are not visible in the bulk but are supported entirely in the strong coupling region? Examples of such instabilities were recently discussed in [46].

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However, the ability to turn on the deformation (2.4) suggests that this is not the case. As we commented above, this deformation prevents the strings from penetrating into the strong coupling region. Furthermore, unlike the examples of [46], where the instability is associated with localized tachyons, it is clear that with (2.4) no localized tachyons can be present.

We can study an alternative quantization in the ϕ-channel. We considered the ground state wave function $\psi_0 \sim e^{-\frac{|m|}{2}|\Phi|^2}$. Alternatively, we might want to consider the wave function $\widetilde{\psi}_0 \sim e^{-\frac{m}{2}|\Phi|^2}$. It is the same as ψ_0 for positive m, but it is not normalizable for negative m. This wave function can be interpreted as associated with a state $|\widetilde{0}\rangle$ which is annihilated by the creation operators of the theory. Its energy is $-|m|$ and acting on it with $U(1)$ invariant combinations of annihilation operators leads to states $|\widetilde{n}\rangle$ with energy $-|m|(2\widetilde{n}+1)$ ($\widetilde{n} = 0, 1, \ldots$). This quantization of the harmonic oscillator leads to a spectrum which is unbounded from below and is also not unitary. Therefore, it is usually ignored. If we follow this strange quantization in the ϕ-channel, we find answers which are compatible with the x-channel picture. One way to understand it is to note that here the energy of $|\widetilde{0}\rangle$ is m rather than $|m|$, and the annihilation operators which lead to lower energy states have energy $-\Gamma_{long} = m = \frac{R}{2} - \frac{1}{2R}$ without absolute value. Then, as for positive m the state $|\widetilde{n}\rangle$ of the ϕ-channel can be interpreted in the x-channel as having $\widetilde{n}$ pairs. Unfortunately, the partition function associated with that quantization diverges. Also, this quantization treats differently positive and negative m, and therefore it does not respect the T-duality of our system.

Finally, we would like to mention some logical options which could invalidate our conclusion about the lack of locality in the x-channel evolution. It might be that our theories are simply inconsistent. Alternatively, they are inconsistent only for some range or R which includes $R = 1$. Finally, perhaps $R = 1$ should be viewed as a limiting radius and a limiting temperature, and the Euclidean circle cannot be reduced below $R = 1$. Clearly, this last possibility is inconsistent with T-duality.

Acknowledgments

It is a pleasure to thank J.L. Davis for collaboration during the early stages of this project. We have also benefited from useful discussions with O. Aharony, N. Itzhaki, I. Klebanov, D. Kutasov, F. Larsen, H. Liu, J. Maldacena and E. Witten. This work was supported in part by grant #DE-FG02-90ER40542.

A. Type 0/II

In this appendix we will reexamine the various compactifications of the type 0 and type II theory of [5] and check whether a modification of the ϕ-channel ground state is needed.

In [5] a moduli space consisting of eight lines of theories was identified. We find that five of the lines do not need any modification. However, in the three lines corresponding to compactifications of the type IIA theory, the naive ϕ-channel vacuum is not invariant. Using the notation of [5] we find:

Line 4. This is a circle compactification of IIA on a circle of radius R. It is dual to IIB on a circle of radius $R' = \frac{2}{R}$. In the IIB language we have a complex periodic fermion of one chirality and a periodic scalar with the opposite chirality. Here the naive ground state has $N = \frac{1}{8}$. This translates in IIA to $W = \frac{1}{8}$. Therefore we might need to change $\Gamma \to \Gamma - \frac{R}{16}$.

Line 6. This is the superaffine IIA theory, and it is selfdual under $R \to \frac{1}{R}$. Here we have an antiperiodic fermion of one chirality and a periodic fermion with the opposite chirality. Therefore, the naive ground state has $N = \frac{1}{16}$, and using T-duality, $W = \frac{1}{8}$. Therefore we might need to change $\Gamma \to \Gamma - \frac{R}{16} - \frac{1}{16R}$.

Line 8. This is a compactification of IIA on a thermal circle of radius R. It is dual to 0B on a twisted circle of radius[7] $R' = \frac{1}{R}$. In the 0B language the left-moving and the right-moving tachyons are anti-periodic. One chirality of the C field is periodic and the other chirality is antiperiodic. Therefore, the naive ground state has $N = \frac{1}{16}$, or in the IIA language $W = \frac{1}{8}$, and we might need to change $\Gamma \to \Gamma - \frac{R}{16}$.

These modifications of the ground state are puzzling for two reasons. First, the theory does not have closed string vertex operators with these quantum numbers. Second, it suggests a modification of the IIA theory with noncompact x by an insertion with nonzero W. Such an insertion violates the parity symmetry of the theory.

We do not have a clear understanding of this puzzle. Perhaps these compactifications are inconsistent. Alternatively, perhaps for some reason no modification is needed here. The most likely possibility is the following. These theories have excitations which cannot be described by worldsheet vertex operators. Such excitations of the type 0 and type II theories were discussed in [47, 5, 19]. Perhaps these excitations can be used to achieve the invariance of the vacuum.

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